Physics 618 2020

March 31, 2020

Important remark I forgot to make:

Central Extensions & Projective Reps.

Suppose we have a proj. rep. of G

Def: of a proj. rep:

$$\rho : G \longrightarrow GL(V)$$

vector space

$$\rho(g_1)\rho(g_2) = \underbrace{\overset{\in U(1)}{c(g_1,g_2)}}_{\text{satisfies cocycle identity}} \rho(g_1 g_2)$$

proj. rep.

$$\boxed{1 \to U(1) \to \widetilde{G} \longrightarrow G \longrightarrow 1}$$

$\widetilde{G}$ is the true symmetry group.

True repn of $\widetilde{G} \ni (z, g)$

$$(z_1, g_1)\cdot(z_2, g_2) = (z_1 z_2\, c(g_1, g_2),\ g_1 g_2)$$

$$\tilde{\rho}(z, g) = z\,\rho(g)$$

$\tilde{\rho}$ TRUE REPn of $\widetilde{G}$.

Weinberg allows a priori

$$\rho(g_1)\,\rho(g_2)\cdot\psi = \underbrace{c(g_1,g_2,\psi)}_{\text{phase.}}\,\rho(g_1g_2)\cdot\psi$$

Weinberg then shows that

$c(g_1,g_2,\psi)$ must be independent of ψ

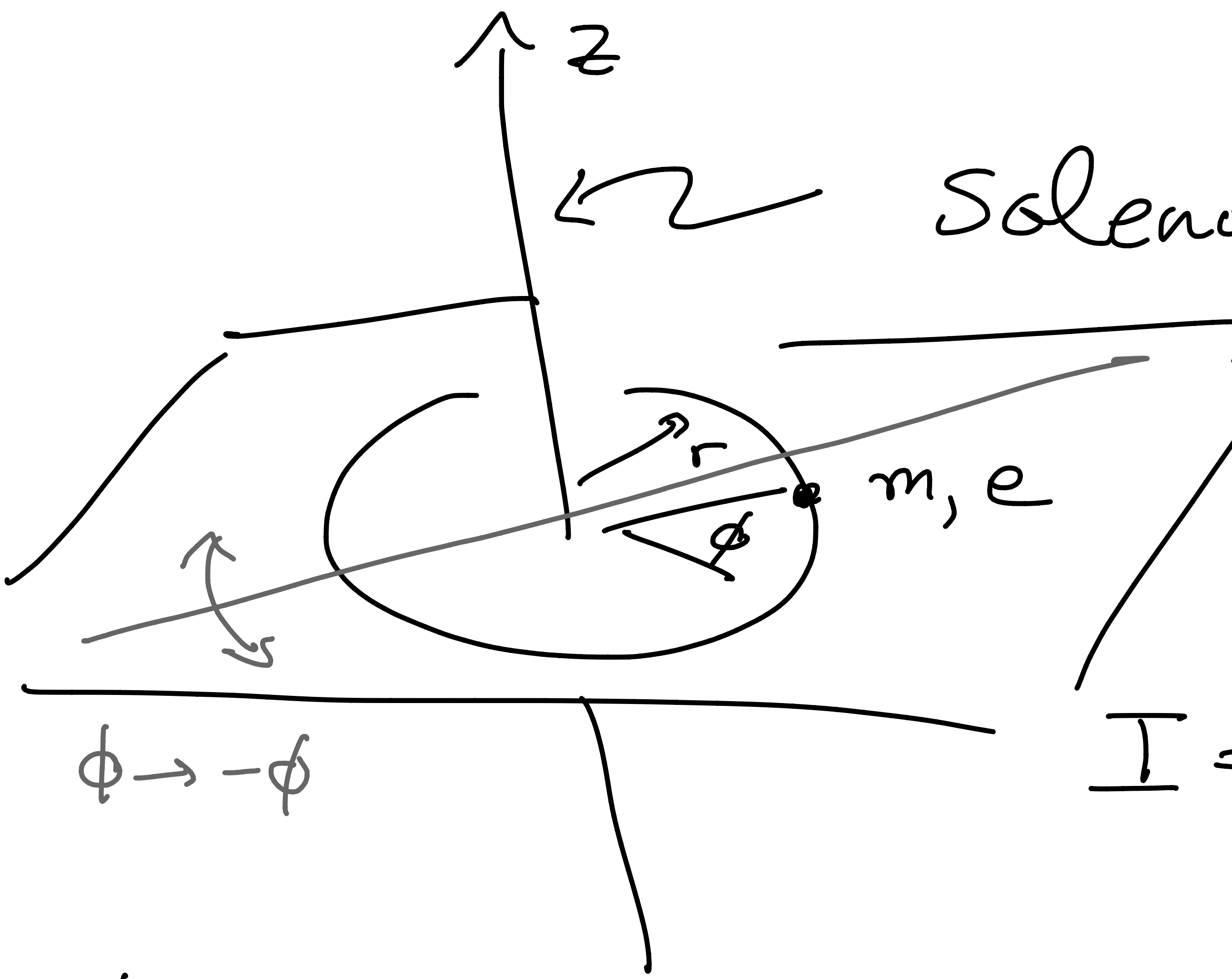
One could take

$$\begin{array}{ccc} V_1 & \oplus & V_2 \\ \uparrow & & \uparrow \end{array}$$

inequivalent projective rep's of G.

$$\begin{array}{ccc} \tilde{G}_1 & & \tilde{G}_2 \end{array}$$

True rep of $\tilde{G}_1 \times \tilde{G}_2$



Solenoid magnetic flux $B_z = B$

$I = mr^2$

group theory
Spectrum
interesting functions
B

$$\boxed{S = \int \frac{1}{2} I \dot{\phi}^2 dt + \underbrace{\frac{eB}{2\pi} \dot{\phi}\, dt}_{\text{does not contribute}}}$$

EOM $\quad \ddot{\phi} = 0$

Classical symmetry group: $G = O(2)$

$P: \quad \phi \longrightarrow -\phi$

$R(\alpha): \phi \longmapsto \phi + \alpha \qquad \alpha \sim \alpha + 2\pi$

$$e^{i\phi} \longrightarrow e^{-i\phi}$$

$$e^{i\phi} \longrightarrow e^{i\alpha} e^{i\phi}$$

In general $\quad A_\mu = (\phi, \vec{A})$

Coul. potential

fieldstrength tensors

$$F_{\mu\nu} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & E_{01} & E_{02} & E_{03} \\ & 0 & B_3 & -B_2 \\ & & 0 & B_1 \\ & & & 0 \end{pmatrix} \end{array}$$

$$F_{0i} = \partial_0 A_i - \partial_i \phi = E_i$$

Sign ?

$$F_{ij} = \tfrac{1}{2} \epsilon_{ijk} B_k$$

$$F_{12} = B_3$$

$$F_{23} = B_1$$

$$F_{13} = -B_2$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P^2 = 1$$

$$R(\alpha) R(\beta) = R(\alpha+\beta)$$

$$P R(\alpha) P = R(-\alpha) = R(\alpha)^{-1}$$

Quantum Theory:

Canonical momentum conj. to ϕ is

$$L := \frac{\delta S}{\delta \dot{\phi}} = I \dot{\phi} + \frac{eB}{2\pi}$$

$$H_{\mathcal{B}} = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial \phi} - \mathcal{B}\right)^2$$

$$\mathcal{B} := \frac{eB}{2\pi\hbar} \in \mathbb{R}$$

1-param. family of Hamiltonians on Hilbert space

$$\mathcal{H} = L^2(S^1).$$

$H_{\mathcal{B}}$ has a complete set of eigenvectors

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad m \in \mathbb{Z}$$

$$H_{\mathcal{B}} \cdot \psi_m = E_m(\mathcal{B})\, \psi_m$$

$$\boxed{E_m(\mathcal{B}) = \frac{\hbar^2}{2I}\left(m - \mathcal{B}\right)^2}$$

For each m only one e.v.

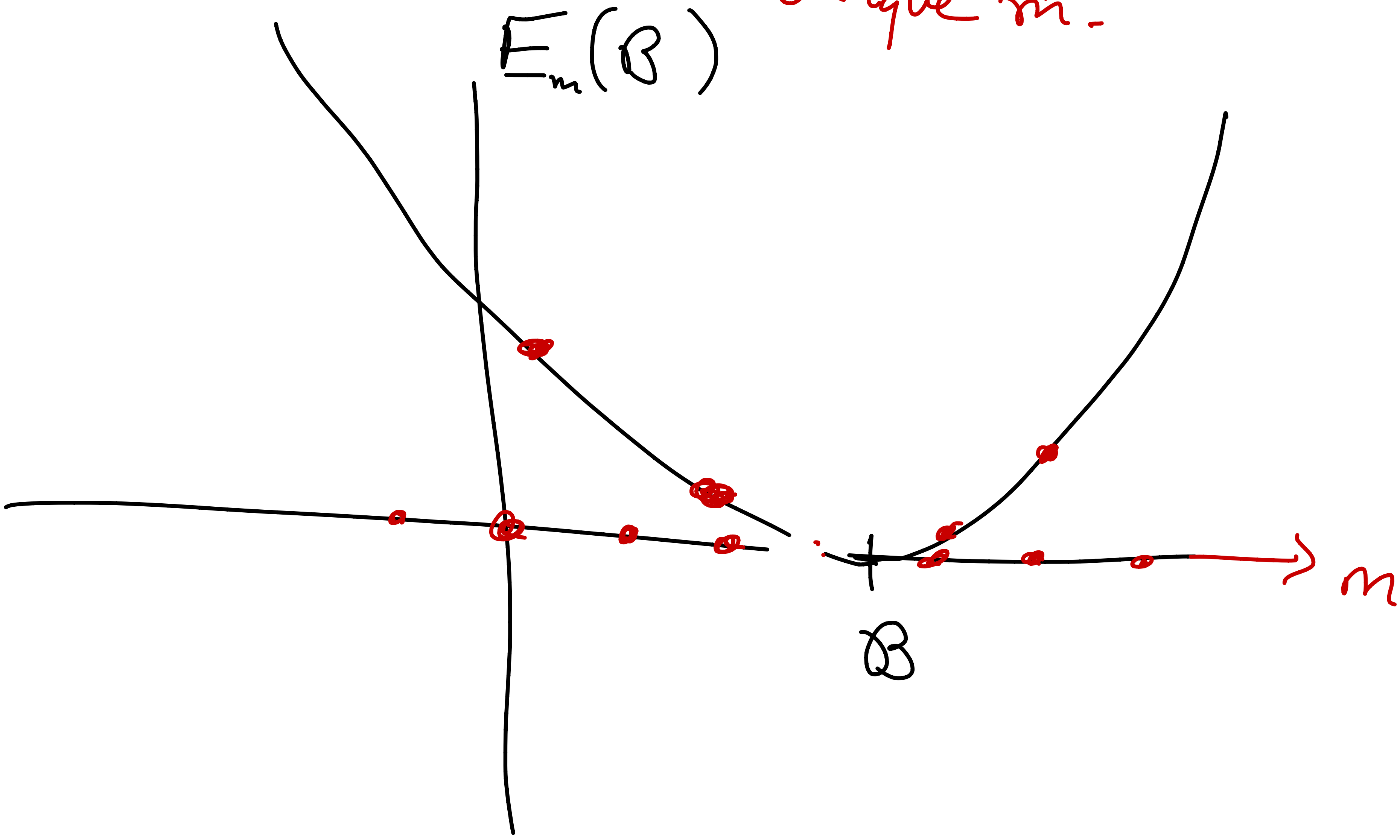
<u>Long list of remarks:</u>

1. Action makes sense $\phi \in \mathbb{R}$ rather than $\phi \sim \phi + 2\pi$. Then m is not quantized and the spectrum of $H_{\mathcal{B}}$ is indpt of $\mathcal{B}$. Topological term has no effect.

2. However $\phi \sim \phi + 2\pi$ single valued position is $e^{i\phi}$ then the topological term matters – even though it is a total derivative.

3. <u>Eigenspaces</u>

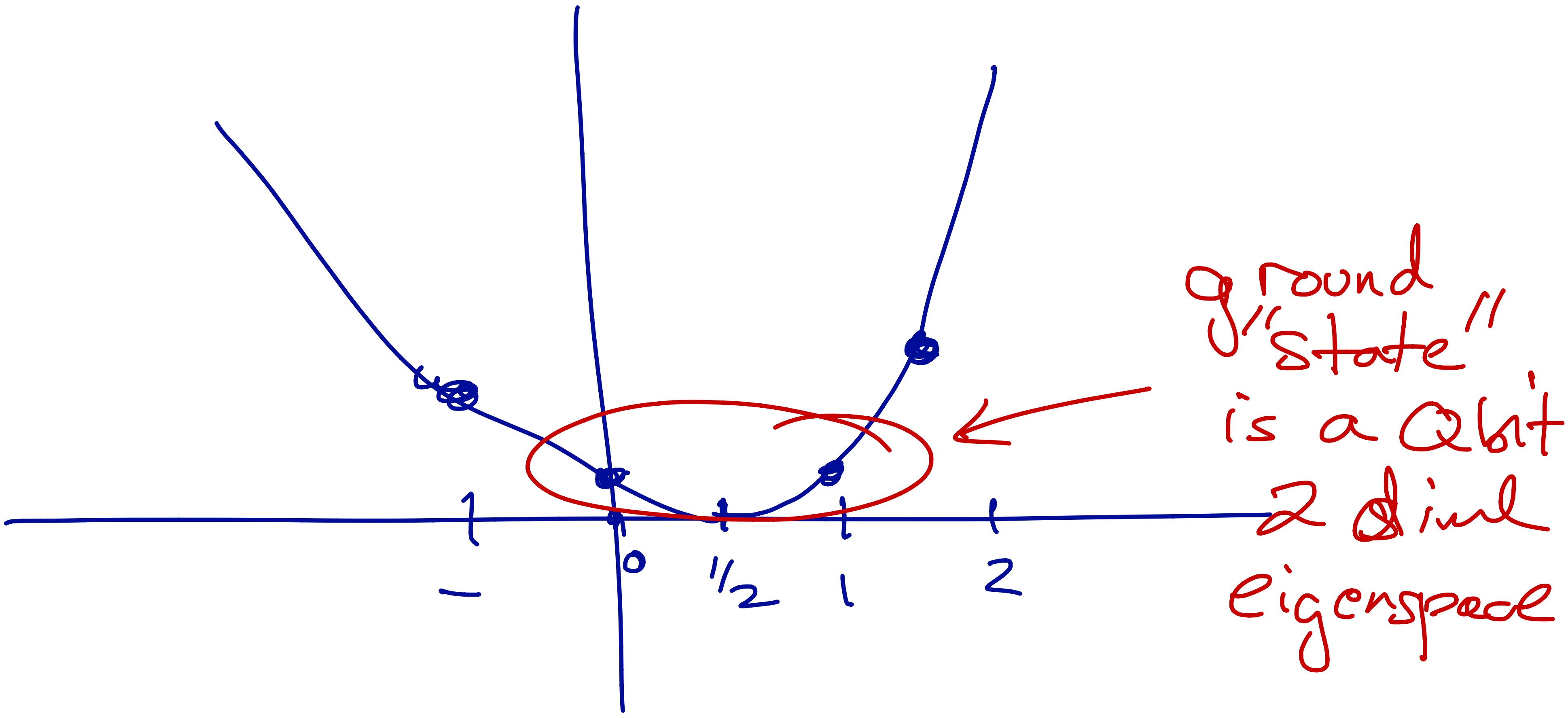
$2B \notin \mathbb{Z}$ all eigenspaces are 1-dim : know energy $\Rightarrow$ unique m.



$$E_m \sim (m-B)^2$$

If $2\mathcal{B} \in \mathbb{Z}$ the energy eigenspaces can have degeneracies.

If $2\mathcal{B}$ odd. e.g. $\mathcal{B} = 1/2$



$$E_m = E_{2\mathcal{B}-m} \qquad 2\mathcal{B} \in \mathbb{Z}$$

all eigenspaces are dim $= 2$.

If $2\mathcal{B}$ is even, all eigenspaces are dim $= 2$ EXCEPT $m = \mathcal{B}$

ground state is 1-dim'l.

4. Spectrum is periodic in $\mathcal{B}$

$$U H_{\mathcal{B}} U^{-1} = H_{\mathcal{B}+1}$$

"physics is periodic in $\mathcal{B}$"

5. Physically realized in mesoscopic systems "Coulomb blockade"

6. <u>This system is a toy field theory</u>

In field theory the "fields" are functions

$$\Phi : M \longrightarrow X$$

M ↑ Spacetime; X ↑ "target space"

$$\mathcal{F} = \text{Space of fields} = \mathrm{Map}(M \to X)$$

$S[\Phi]$ — action

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e.g. nonrelativistic Particle moving on a Riemannian manifold:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$$

$$S = \int \frac{m}{2} g_{\mu\nu}(x(t)) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt$$

$$\mathcal{F}: \quad \mathrm{Map}(M \longrightarrow X)$$

M ↑ time

$M = \mathbb{R}$

$M = [t_{in}, t_{fin}]$

$M = S^1$

Generalizes further M is $d+1$ dimensional with Lorentzian metric

$$ds^2_M = h_{ab}(\sigma)\, d\sigma^a d\sigma^b$$

$$\sigma^a \qquad a = \overset{\text{time}}{\downarrow}{0}, \underbrace{1, \dots, d}_{\text{space}}$$

$$ds^2_X = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$$\int_M \frac{m}{2} h^{ab}(\sigma)\, g_{\mu\nu}(x(\sigma))\, \partial_a x^\mu \partial_b x^\nu \sqrt{|\det h_{ab}|}\, d^{d+1}\sigma$$

Nonlinear σ-model.

$$\mathcal{F} = \mathrm{Map}(M \longrightarrow X)$$

Q.M. of a particle is a $0+1$ dim'l field theory. ($M = 1+1$ dim'l is basis for string theory.)

If in addition X has a gauge field on it, and particle has charge e then action

$$S = \int \frac{1}{2} m \, g_{\mu\nu}(x(t)) \, \dot{x}^\mu \dot{x}^\nu \, dt$$

$$+ e A_\mu(x(t)) \, \dot{x}^\mu \, dt$$

Our case: field is a map

$$e^{i\phi} : \underset{\mathbb{R}}{\overset{M}{\shortparallel}} \longrightarrow S^1 = \text{unit complex \#'s in } \mathbb{C}.$$

$\phi \to -\phi$ "parity" — particle on ring viewpoint

Charge conjugation:

$$\left. e^{i\phi} \longrightarrow (e^{i\phi})^* \right\} \text{ field theory viewpoint.}$$

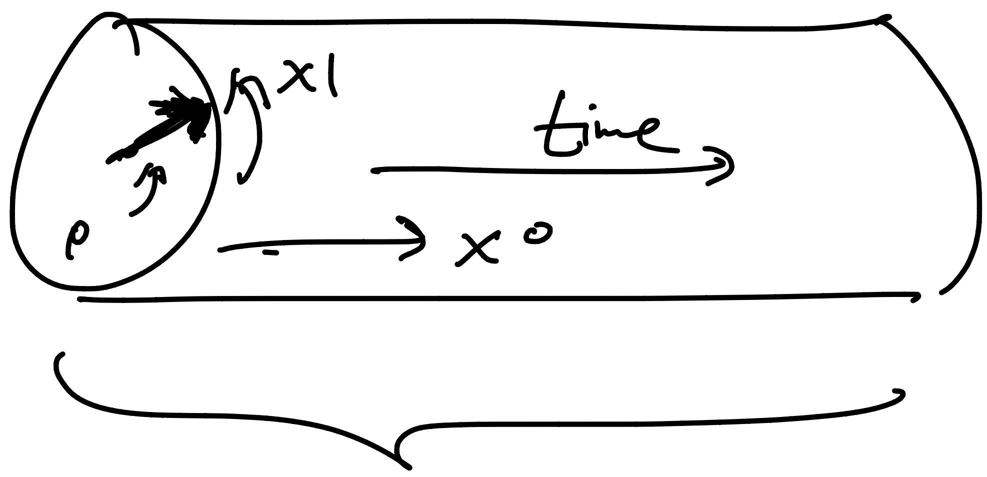
Also there are world volume symmetries

$t \to t + t_0$ and $t \to -t$

We might talk about these later

7. θ - terms in other field theories

Maxwell's theory in 1+1 dim's on a cylinder



M

$A_\mu \quad \mu = 0, 1$

$F_{\mu\nu}$ = only $F_{01} = -F_{10}$ can be nonzero

$$S_{Max} = \int \frac{1}{2e^2} dx^0 dx^1 \, F_{01}^2$$

$$F_{01}^2 = E^2$$

$$E^2 - \cancel{B^2}$$

but we can add a topological term:

"axion electrodynamics" $dx^0 dx^1$

$$S = \int \frac{1}{2e^2} F_{01}^2 \, dx^0 dx^1 + \int \frac{\theta}{2\pi} F_{01} \, dx^0 \wedge dx^1$$

$$F_{01} = \partial_0 A_1 - \partial_1 A_0$$

Choose $A_0 = 0$ gauge

In KK reduction

$$A_1 = \sum_{n \in \mathbb{Z}} e^{2\pi i n x^1 / \rho} A_1^{(n)}$$

Working with the zeroth fourier coeff:

$$e^{i\phi(t)} = e^{i \oint_{S^1} A} = e^{i \oint_{S^1} A_1 dx^1}$$

$\Rightarrow$ get the above Quantum System $\quad I \sim e^2 \hbar$

$$\boxed{\theta = 2\pi B}$$

Differential forms : Proper mathematical formalism for discussing gauge theories

$$F = \frac{1}{2!} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$\left.\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned}\right\} \text{Maxwell's eqs. (in vacuum)}$$

There is also an important topological term in 3+1 gauge theory

Maxwell case: $U(1)$ gauge theory

$$S' = \int d^4x \, \underbrace{\frac{1}{4e^2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)}_{\sim(\vec{E}^2 - \vec{B}^2)} + \int \frac{\theta}{8\pi} \epsilon^{\mu\nu\lambda\rho} \underbrace{\mathrm{tr}\left(F_{\mu\nu} F_{\lambda\rho} d^4x\right)}_{\vec{E} \cdot \vec{B}}$$

Show: this is a total derivative!

Effective theory of electromagnetic fields in materials such θ- terms appear.

If we have P- inv. or Time Rev. inv.

$\theta \to -\theta$ and $\theta \sim \theta + 2\pi$

(Remember "physics is periodic in θ")

$\theta = 0, \pi$

if Material is an insulator

$\theta = 0$ "normal insulator"

$\theta = \pi$ "topological insulator"

2B is an integer

If we consider nonabelian gauge fields

$$\underbrace{SU(3)}_{\text{strong}} \times \underbrace{SU(2) \times U(1)}_{\text{electroweak}}$$

Topological terms matter and $\theta_{SU(3)}$ induces an intrinsic electric dipole moment for the neutron: linear in θ

Experimentally $|\theta_{SU(3)}| < 10^{-9}$

!!!!!

Strong CP- Problem

"Because of principle of naturalness" if no symmetry reason excludes a term in action it should be there

Field space space of connections on a principal bundle $\mathcal{A}/\mathcal{G}$ has nontrivial topology: So fieldspace has nontrivial topology and topological terms matter.

Back to particle on the ring

Wigner tells us that some extension of $O(2)$ by $U(1)$ should act on Hilbert space commuting with Hamiltonian.

$$R(\alpha): \quad \phi \longrightarrow \phi + \alpha$$

$$\Psi_m \sim e^{im\phi} \longrightarrow e^{im\alpha} e^{im\phi}$$

$$\boxed{R(\alpha) \cdot \Psi_m = e^{im\alpha} \Psi_m}$$

$$\rho(R(\alpha)) = \mathcal{R}(\alpha)$$

$\rho(P) = \mathcal{P}$ $\qquad P: \phi \to -\phi$

$$e^{im\phi} \longrightarrow e^{-im\phi}$$

does not commute with Hamiltonian

$$H_{\mathcal{B}} = \frac{1}{2I}\left(-i\hbar\frac{\partial}{\partial\phi} - \mathcal{B}\right)^2$$

$\mathcal{B} \neq 0 \qquad \phi \to -\phi \qquad \mathcal{B} \to -\mathcal{B}$

$H_{\mathcal{B}}$ is not unitarily equiv. to $H_{-\mathcal{B}}$

$$\mathcal{P} \cdot \Psi_m = \xi_m \Psi_{2\mathcal{B}-m}$$

phase: but it will not matter - just a distraction so put $\xi_m = 1$.

If $2\mathcal{B} \in \mathbb{Z}$.

$\boxed{\text{If } 2\mathcal{B} \notin \mathbb{Z} \quad O(2) \text{ is broken by g.s. to } SO(2).}$

If $2\mathcal{B} \in \mathbb{Z}$ then

$$\mathcal{P} \cdot \psi_m := \psi_{2\mathcal{B}-m}$$

implements parity / charge conj. and commutes with $H_{\mathcal{B}}$.

Recall the classical operator relations:

$$P^2 = 1$$

$$R(\alpha) R(\beta) = R(\alpha+\beta)$$

$$P R(\alpha) P^{-1} = R(-\alpha) = R(\alpha)^{-1}$$

What about the quantum op's?

$$\mathcal{P}^2 = 1 \quad \checkmark$$

$$\mathcal{R}(\alpha) \mathcal{R}(\beta) = \mathcal{R}(\alpha+\beta) \quad \checkmark$$

But — exercise! — you check

$$\boxed{P\,R(\alpha)\,P = \underbrace{e^{i(2B)\alpha}}\,R(-\alpha)}$$

<u>Note:</u> equation only makes sense when $2B \in \mathbb{Z}$.

Now consider the group of operators generated by

$$\boxed{P,\ R(\alpha),\ z\,\mathbb{1}_{\mathcal{H}}} \qquad z \in U(1)$$

How is this related to the naive answer $U(1) \times O(2)$?

Not the relations of

$$O(2) = SO(2) \rtimes \mathbb{Z}_2$$

$$\mathrm{Aut}(SO(2)) \leftarrow \sigma$$

$$\sigma: R(\alpha) \to R(-\alpha)$$